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# HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGICALLY FIBERED KNOTS

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## 1. Introduction

This note is adapted from the talk at the 2010 Intelligence of Low-dimensional Topology at Research Institute for Mathematical Sciences, Kyoto University. For the detail, see the original papers [12], [13].

Let $\Sigma_{g,n}$ be a compact oriented surface of genus $g$ with $n \geq 1$ boundary components, and the triple $(M, i_+, i_-)$ be an oriented homology cobordism between $\Sigma_{g,n}$ and $\Sigma_{g,n}$ with two markings of $\partial M$ : $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$. We call $(M, i_+, i_-)$ a *homology cylinder* over $\Sigma_{g,n}$. This object was introduced by Goussarov [14] and Habiro [16] since it is suitable for applying the theory of clovers and claspers, and then has been studied together with finite type invariants of 3-manifolds. The following have been known as methods for constructing homology cylinders:

- connected sums of the trivial cobordism with homology 3-spheres;
- Levine's method [19] using string links in the 3-ball;
- Habegger's method [15] giving homology cylinders as results of surgeries along string links in homology 3-balls; and
- clasper surgeries (see [14] and [16]).

In [12], the authors gave an explicit construction of homology cylinders, i.e. we introduced a notion of a *homologically fibered knot* and construct a homology cylinder using it. The family of the homologically fibered knots include that of the fibered knots. So, roughly speaking, the following relationships exist:

$$\begin{array}{ccccc} \text{Pure Braid} & \longleftrightarrow & \text{Mapping cylinder} & \longleftrightarrow & \text{Fibered knot} \\ \cap & & \cap & & \cap \\ \text{Pure String link} & \overset{\text{Levine}}{\longleftrightarrow} & \text{Homology cylinder} & \longleftrightarrow & \text{Homologically fibered knot} \\ \text{(Habegger-Lin)} & & \text{(Goussarov, Habiro)} & & \end{array}$$

In [18], Kirk-Livingston-Wang introduced a Reidemeister torsion for string links, then the second author studied the corresponding Reidemeister torsion for homology cylinders in [23]. Note that this torsion may be regarded as a special case of a decatogorification of sutured Floer homology [8]. In this note, we study the Reidemeister torsion for homologically fibered knots and show a factorization formula. Further, we give a MATHEMATICA program for explicit calculations of the invariants for homologically fibered knots.

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## 2. Homologically fibered knots

In this section, we introduce two main objects in this note: homology cylinders and sutured manifolds. First, we define homology cylinders over surfaces, which have their origin in Goussarov [14], Habiro [16], Garoufalidis-Levine [11] and Levine [19]. Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components.

**Definition 2.1.** A *homology cylinder* $(M, i_+, i_-)$ over $\Sigma_{g,n}$ consists of a compact oriented 3-manifold $M$ with two embeddings $i_+, i_- : \Sigma_{g,n} \hookrightarrow \partial M$ such that:

(i) $i_+$ is orientation-preserving and $i_-$ is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,n}) \cup i_-(\Sigma_{g,n})$ and $i_+(\Sigma_{g,n}) \cap i_-(\Sigma_{g,n}) = i_+(\partial\Sigma_{g,n}) = i_-(\partial\Sigma_{g,n})$;
(iii) $i_+|_{\partial\Sigma_{g,n}} = i_-|_{\partial\Sigma_{g,n}}$; and
(iv) $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

If we replace (iv) with the condition that $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Q}) \to H_*(M; \mathbb{Q})$ are isomorphisms, then $(M, i_+, i_-)$ is called a *rational homology cylinder*.

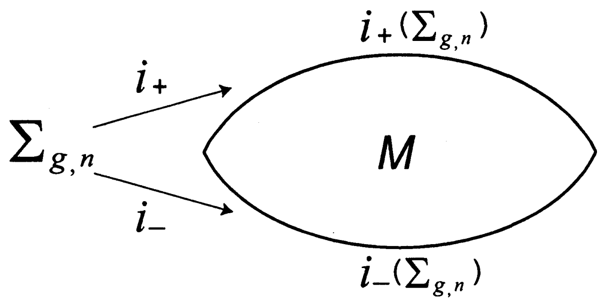


Figure 1. Homology cylinder

We often write a (rational) homology cylinder $(M, i_+, i_-)$ briefly by $M$. Note that our definition is the same as that in [11] and [19] except that we may consider homology cylinders over surfaces with multiple boundaries.

Two (rational) homology cylinders $(M, i_+, i_-)$ and $(N, j_+, j_-)$ over $\Sigma_{g,n}$ are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism $f : M \xrightarrow{\cong} N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$. We denote the set of isomorphism classes of homology cylinders (resp. rational homology cylinders) over $\Sigma_{g,n}$ by $\mathcal{C}_{g,n}$ (resp. $\mathcal{C}^{\mathbb{Q}}_{g,n}$).

**Example 2.2** (Mapping cylinder). For each diffeomorphism $\varphi$ of $\Sigma_{g,n}$ which fixes $\partial\Sigma_{g,n}$ pointwise (hence, $\varphi$ preserves the orientation of $\Sigma_{g,n}$), we can construct a homology cylinder by setting

$$(\Sigma_{g,n} \times [0,1], \mathrm{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,n})$ and $i_-(\Sigma_{g,n})$ are stretched half-way along $(\partial\Sigma_{g,n}) \times [0,1]$. It is easily checked that the isomorphism class of $(\Sigma_{g,n} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$ depends only on the (boundary fixing) isotopy class of $\varphi$. Therefore, this construction gives a map from the mapping class group $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ to $\mathcal{C}_{g,n}$.

Next, we recall the definition of sutured manifolds given by Gabai [10]. We here use a special case of them.

A *sutured manifold* $(M, \gamma)$ is a compact oriented 3-manifold $M$ together with a subset $\gamma \subset \partial M$ which is a union of finitely many mutually disjoint annuli. For each component of $\gamma$, an oriented core circle called a *suture* is fixed, and we denote the set of sutures by $s(\gamma)$. Every component of $R(\gamma) = \partial M - \operatorname{Int}\gamma$ is oriented so that the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$, that is, the orientation of each component of $\partial R(\gamma)$ induced from that of $R(\gamma)$ is parallel to the orientation of the corresponding component of $s(\gamma)$. We denote by $R_+(\gamma)$ (resp. $R_-(\gamma)$) the union of those components of $R(\gamma)$ whose normal vectors point out of (resp. into) $M$.

**Example 2.3.** For a knot $K$ in $S^3$ and a Seifert surface $\bar{R}$ of $K$, we set $R := \bar{R} \cap E(K)$, called also a Seifert surface, where $E(K) = \overline{S^3 - N(K)}$ is the complement of a regular neighborhood $N(K)$ of $K$. Then $(M_R, \gamma) := (\overline{E(K) - N(R)}, \overline{\partial E(K) - N(\partial R)})$ defines a sutured manifold. We call it the *complementary sutured manifold* for $R$. In this paper, we simply call it the sutured manifold for $R$.

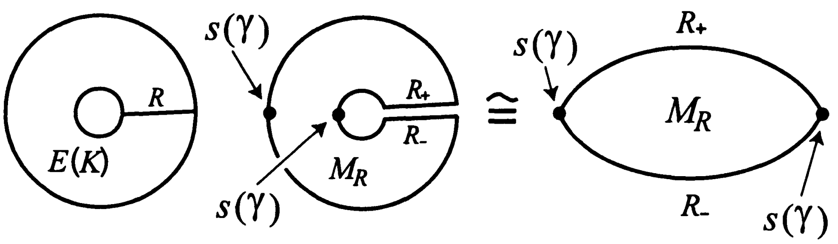


FIGURE 2. Complementary sutured manifold

Let $L$ be an oriented link in the 3-sphere $S^3$, and $\Delta_L(t)$ the normalized (one variable) Alexander polynomial of $L$, i.e. the lowest degree of $\Delta_L(t)$ is 0.

**Definition 2.4.** An $n$-component link $L$ in $S^3$ is said to be *homologically fibered* if $L$ satisfies the following two conditions:

(i) The degree of $\Delta_L(t)$ is $2g + n - 1$, where $g$ is the genus of a connected Seifert surface of $L$; and
(ii) $\Delta_L(0) = \pm 1$.

If an $n$-component link $L$ satisfies (i), then $L$ is said to be *rationally homologically fibered*.

The Alexander polynomial that satisfies the condition (ii) is said to be *monic* in this paper.

**Remark 2.5.** In general, if $L$ bounds a connected Seifert surface of genus $g$, then

$$2g + n - 1 \geq (\text{the degree of } \Delta_L(t)).$$

It is known ([5], [21]) that if $L$ has an alternating diagram that gives, by the Seifert algorithm, a connected Seifert surface of genus $g$, then the degree of $\Delta_L(t)$ is equal to $2g + n - 1$.

**Remark 2.6.** Suppose $L$ is an alternating link. Then, $L$ is fibered if and only if $\Delta_L(t)$ is monic, by Murasugi [22] (see also 13.26 (c) in [1]). Therefore, if a homologically fibered link $L$ is not fibered, then $L$ is non-alternating.

Let $L$ be an $n$-component link and $\Sigma_{g,n}$ the compact oriented surface that is diffeomorphic to a Seifert surface $R$ of $L$. We fix a diffeomorphism $\vartheta : \Sigma_{g,n} \xrightarrow{\cong} R$ and denote by $(M_R, \gamma)$ the complementary sutured manifold for $R$. Then we may see that there are an orientation-preserving embedding $i_+ : \Sigma_{g,n} \to M_R$ and an orientation-reversing embedding $i_- : \Sigma_{g,n} \to M_R$ with $i_+(\Sigma_{g,n}) = R_+(\gamma)$ and $i_-(\Sigma_{g,n}) = R_-(\gamma)$, where two embeddings $i_\pm$ are the composite mappings of $\vartheta$ and embeddings $\iota_\pm : R \hookrightarrow M_R$ such that $i_\pm = \iota_\pm \circ \vartheta : \Sigma_{g,n} \to R_\pm(\gamma) \subset M_R$:

$$\begin{tikzcd} \Sigma_{g,n} \arrow[r, "\vartheta"] \arrow[dr, "i_\pm"'] & R \arrow[d, "\iota_\pm"] \\ & M_R \end{tikzcd}$$

If $i_+, i_- : H_1(\Sigma_{g,n}) \to H_1(M_R)$ are isomorphisms, we may regard $(M_R, \gamma)$ as a homology cylinder. The next proposition was essentially mentioned in [6]. A proof is given in [12].

**Proposition 2.7.** *Let $R$ be a Seifert surface of a link $L$. If the complementary sutured manifold for $R$ is a homology cylinder, then $L$ is homologically fibered. Conversely, if $L$ is homologically fibered, then the complementary sutured manifold for each minimal genus Seifert surface of $L$ is a homology cylinder.*

It is known that all homologically fibered knots are fibered among prime knots with at most 11 crossings. On the other hand, Friedl-Kim [9] (see also [2]) showed that there are 13 non-fibered homologically fibered knots with 12-crossings. See Figure 7.

## 3. Factorization formulas of Alexander invariants

Let $R$ be a minimal genus Seifert surface of a rationally homologically fibered knot $K$ in $S^3$, and $M_R$ be the sutured manifold for $R \cong \Sigma_{g,1}$. We fix a basis of $H_1(R;\mathbb{Q})$, which yields an isomorphism $H_1(R;\mathbb{Q}) \cong \mathbb{Q}^{2g}$. Then we can rewrite the definition $\Delta_K(t) = \det(S - tS^T)$ of the Alexander polynomial of $K$ by using the invertibility (over $\mathbb{Q}$) of the Seifert matrix $S$, and obtain a factorization

$$\Delta_K(t) = \det(S)\det(I_{2g} - t\sigma(M_R)) \tag{3.1}$$

of $\Delta_K(t)$. Note that $\sigma(M_R) := S^{-1}S^T$ represents the composite of isomorphisms

$$\mathbb{Q}^{2g} \cong H_1(R;\mathbb{Q}) \xrightarrow[i_-]{\cong} H_1(M_R;\mathbb{Q}) \xrightarrow[i_+^{-1}]{\cong} H_1(R;\mathbb{Q}) \cong \mathbb{Q}^{2g}.$$

The matrix $\sigma(M_R)$ can be interpreted as a monodromy of $M_R$ from a view point of the rational homology. Regarding the formula (3.1) as a basic case, we constructed in [12] its generalization under the framework of *higher-order Alexander invariants* due to Cochran [3], Harvey [17] and Friedl [7]. In this procedure, the Seifert matrix $S$, the monodromy $\sigma(M_R)$ and $\Delta_K(t)$ are generalized to a certain Reidemeister torsion $\tau_\rho^+(M_R)$, the Magnus matrix $r_\rho(M_R)$ and some higher-order (non-commutative) Reidemeister torsion $\tau_\rho(E(K))$ associated with a representation $\rho$ of the fundamental group of $M_R$.

Here, we review higher-order Alexander invariants quickly. For a matrix $A$ with entries in a group ring $\mathbb{Z}G$ (or its quotient field) for a group $G$, we denote by $\overline{A}$ the matrix obtained from $A$ by applying the involution induced from $(x \mapsto x^{-1},\ x \in G)$ to each entry. For a module $M$, we write $M^n$ for the module of column vectors with $n$ entries. For a finite cell complex $X$, we denote by $\widetilde{X}$ its universal covering. We take a base point $p$ of $X$ and a lift $\widetilde{p}$ of $p$ as a base point of $\widetilde{X}$. $\pi := \pi_1(X, p)$ acts on $\widetilde{X}$ from the *right* through its deck transformation group, so that the lift of a loop $l \in \pi$ starting from $\widetilde{p}$ reaches $\widetilde{p}l^{-1}$. Then the cellular chain complex $C_*(\widetilde{X})$ of $\widetilde{X}$ becomes a right $\mathbb{Z}\pi$-module. For each left $\mathbb{Z}\pi$-algebra $\mathcal{R}$, the twisted chain complex $C_*(X;\mathcal{R})$ is given by the tensor product of the right $\mathbb{Z}\pi$-module $C_*(\widetilde{X})$ and the left $\mathbb{Z}\pi$-module $\mathcal{R}$, so that $C_*(X;\mathcal{R})$ and $H_*(X;\mathcal{R})$ are right $\mathcal{R}$-modules.

In the definition of higher-order Alexander invariants, PTFA groups play important roles, where a group $\Gamma$ is said to be *poly-torsion-free abelian* (*PTFA*) if it has a sequence

$$\Gamma = \Gamma_0 \triangleright \Gamma_1 \triangleright \cdots \triangleright \Gamma_n = \{1\}$$

whose successive quotients $\Gamma_i/\Gamma_{i+1}$ $(i \geq 0)$ are all torsion-free abelian. An advantage of using PTFA groups is that the group ring $\mathbb{Z}\Gamma$ (or $\mathbb{Q}\Gamma$) of $\Gamma$ is known to be an *Ore domain* so that it can be embed into the field (skew field in general)

$$\mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1} = \mathbb{Q}\Gamma(\mathbb{Q}\Gamma - \{0\})^{-1}$$

called the *right field of fractions.* A typical example of PTFA groups is $\mathbb{Z}^n$, where $\mathcal{K}_{\mathbb{Z}^n}$ is isomorphic to the field of rational functions with $n$ variables.

For a rationally homologically fibered knot $K$, we take a homomorphism $\rho : G(K) := \pi_1(E(K)) \to \Gamma$ whose target $\Gamma$ is PTFA. We suppose that $\rho$ is non-trivial. We regard $\mathcal{K}_\Gamma$ as a local coefficient system on $E(K)$ through $\rho$.

**Lemma 3.1** (Cochran [3, Lemma 3.9]). *For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group $\Gamma$, we have $H_*(E(K);\mathcal{K}_\Gamma) = 0$.*

By this lemma, we can define the Reidemeister torsion

$$\tau_\rho(E(K)) := \tau(C_*(E(K);\mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/\pm\rho(G(K))$$

for the acyclic complex $C_*(E(K);\mathcal{K}_\Gamma)$. We refer to Milnor [20] for generalities of torsions. By higher-order Alexander invariants for $K$, we here mean this torsion $\tau_\rho(E(K))$.

We now describe a factorization of $\tau_\rho(E(K))$ generalizing (3.1). Let $(M_R, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathbb{Q}}$ be the rational homology cylinder obtained as the sutured manifold for a minimal genus Seifert surface $R$ of $K$. We use the same notation $\rho : \pi_1(M_R) \to \Gamma$ for the composition $\pi_1(M_R) \to G(K) \xrightarrow{\rho} \Gamma$. Applying Cochran-Orr-Teichner [4, Proposition 2.10], we have the following:

**Lemma 3.2.** *$i_+, i_- : H_*(\Sigma_{g,1}, p; i_\pm^*\mathcal{K}_\Gamma) \to H_*(M_R, p; \mathcal{K}_\Gamma)$ are isomorphisms as right $\mathcal{K}_\Gamma$-vector spaces. Equivalently, $H_*(M_R, i_\pm(\Sigma_{g,1});\mathcal{K}_\Gamma) = 0$.*

This lemma provides the following two kinds of invariants for $M_R$.

**The Magnus matrix** Let $X \subset \Sigma_{g,1}$ be the bouquet of $2g$ circles $\gamma_1, \dots, \gamma_{2g}$ tied at $p$ (see Figure 3). $X$ is a deformation retract of $\Sigma_{g,1}$ relative to $p$. Therefore, for $\pm \in \{+,-\}$, we have

$$H_1(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_\Gamma) \cong H_1(X, p; i_\pm^* \mathcal{K}_\Gamma) = C_1(\widetilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_\pm^* \mathcal{K}_\Gamma \cong \mathcal{K}_\Gamma^{2g}$$

with a basis

$$\{\widetilde{\gamma}_1 \otimes 1, \dots, \widetilde{\gamma}_{2g} \otimes 1\} \subset C_1(\widetilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_\pm^* \mathcal{K}_\Gamma$$

as a right $\mathcal{K}_\Gamma$-vector space. Here we fix a lift $\widetilde{p}$ of $p$ as a base point of $\widetilde{X}$, and denote by $\widetilde{\gamma}_i$ the lift of the oriented loop $\gamma_i$ starting from $\widetilde{p}$.

**Definition 3.3.** For $M_R = (M_R, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathbb{Q}}$, the *Magnus matrix*

$$r_\rho(M_R) \in GL(2g, \mathcal{K}_\Gamma)$$

of $M_R$ is defined as the representation matrix of the right $\mathcal{K}_\Gamma$-isomorphism

$$\mathcal{K}_\Gamma^{2g} \cong H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \xrightarrow[i_-]{\cong} H_1(M_R, p; \mathcal{K}_\Gamma) \xrightarrow[i_+^{-1}]{\cong} H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \cong \mathcal{K}_\Gamma^{2g},$$

where the first and the last isomorphisms use the bases mentioned above.

The matrix $r_\rho(M_R)$ can be interpreted as a monodromy of $M_R$ from a view point of the twisted homology with coefficients in $\mathcal{K}_\Gamma$.

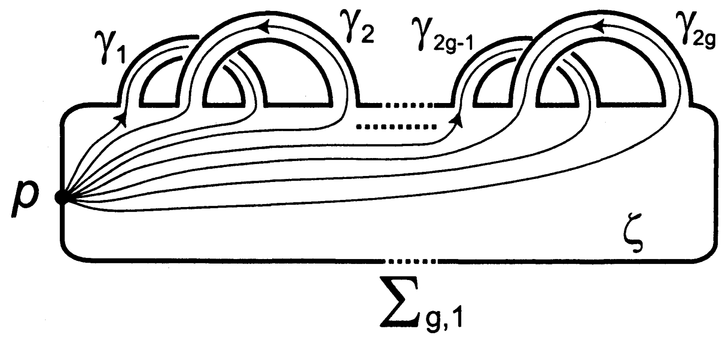


FIGURE 3. Cell decomposition of $\Sigma_{g,1}$

**Γ-torsion** Since the relative complex $C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)$ obtained from any cell decomposition of $(M_R, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 3.2, we can define the following:

**Definition 3.4.** For $M_R = (M_R, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathbb{Q}}$, the Γ-*torsion* $\tau_\rho^+(M_R)$ of $M_R$ is defined by

$$\tau_\rho^+(M_R) := \tau(C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/\pm\rho(\pi_1(M_R)).$$

A method for computing $r_\rho(M_R)$ and $\tau_\rho^+(M_R)$ is given in [12, Section 4], which is based on Kirk-Livingston-Wang's method [18] for invariants of string links, and we now recall it briefly. An *admissible presentation* of $\pi_1(M_R)$ is defined to be the one of the form

$$\langle i_-(\gamma_1), \dots, i_-(\gamma_{2g}), z_1, \dots, z_l, i_+(\gamma_1), \dots, i_+(\gamma_{2g}) \mid r_1, \dots, r_{2g+l} \rangle \tag{3.2}$$

for some integer $l$. That is, it is a finite presentation with deficiency $2g$ whose generating set contains $i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), i_+(\gamma_1), \ldots, i_+(\gamma_{2g})$ and is ordered as above. Such a presentation always exists. For any admissible presentation, define $2g \times (2g+l)$, $l \times (2g+l)$ and $2g \times (2g+l)$ matrices $A, B, C$ over $\mathbb{Z}\Gamma$ by

$$A = \overline{{}^{\rho}\left(\frac{\partial r_j}{\partial i_-(\gamma_i)}\right)}_{\substack{1\le i\le 2g\\ 1\le j\le 2g+l}}, \quad B = \overline{{}^{\rho}\left(\frac{\partial r_j}{\partial z_i}\right)}_{\substack{1\le i\le l\\ 1\le j\le 2g+l}}, \quad C = \overline{{}^{\rho}\left(\frac{\partial r_j}{\partial i_+(\gamma_i)}\right)}_{\substack{1\le i\le 2g\\ 1\le j\le 2g+l}}$$

**Proposition 3.5** ([12, Propositions 4.5, 4.6]). *As matrices with entries in $\mathcal{K}_\Gamma$, we have:*

(1) *The square matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is invertible and $\tau_\rho^+(M_R) = \begin{pmatrix} A \\ B \end{pmatrix}$; and*

(2) $r_\rho(M_R) = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ O_{(l,2g)} \end{pmatrix}$

Using the above invariants, the factorization formula for $\tau_\rho(E(K))$ is given as follows:

**Theorem 3.6.** *Let $K$ be a rationally homologically fibered knot of genus $g$. For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group $\Gamma$, a loop $\mu$ representing the meridian of $K$ satisfies $\rho(\mu) \neq 1 \in \Gamma \subset \mathcal{K}_\Gamma$ and we have a factorization*

$$\tau_\rho(E(K)) = \frac{\tau_\rho^+(M_R) \cdot (I_{2g} - \rho(\mu) r_\rho(M_R))}{1 - \rho(\mu)} \in K_1(\mathcal{K}_\Gamma)/\pm \rho(G(K)) \tag{3.3}$$

*of the torsion $\tau_\rho(E(K))$.*

To compare (3.3) with (3.1), recall Milnor's formula [20] that $\dfrac{\Delta_K(t)}{1-t}$ represents the Reidemeister torsion associated with the abelianization map $\rho_1 : G(K) \to \langle t \rangle \subset \mathbb{Q}(t)$. Taking $\rho_1$ as $\rho$, we recover the formula (3.1).

## 4. Computations

Although all the ingredients in the formula (3.3) are theoretically determined by information on fundamental groups, it is difficult to compute them explicitly because of the non-commutativity of $\mathcal{K}_\Gamma$ except in some special cases including the following.

Let $K$ be a homologically fibered knot with a minimal genus Seifert surface $R$ and let $M_R$ be the sutured manifold for $R$. Consider the group extension

$$1 \longrightarrow G(K)'/G(K)'' \longrightarrow D_2(K) \longrightarrow G(K)/G(K)' = H_1(E(K)) \cong \mathbb{Z} \longrightarrow 1 \tag{4.1}$$

relating to the metabelian quotient $D_2(K) := G(K)/G(K)''$ of $G(K)$. We have

$$G(K)'/G(K)'' \cong H_1(R) \cong H_1(M_R)$$

since it coincides with the first homology of the infinite cyclic covering of $E(K)$, which can be seen as the product of infinitely many copies of $M_R$. In particular, we may regard $H_1(M_R)$ as a natural (namely, independent of choices of minimal genus Seifert surfaces) subgroup of $D_2(K)$. We take $\rho$ to be the natural projection

$$\rho_2 : G(K) \longrightarrow D_2(K).$$

It is known that $D_2(K)$ is PTFA, so that $\mathcal{K}_{D_2(K)}$ is defined. Then, Proposition 3.5 shows that $\tau^+_{\rho_2}(M_R)$ and $r_{\rho_2}(M_R)$ can be computed by calculations on a commutative subfield $\mathcal{K}_{H_1(M_R)}$ of $\mathcal{K}_{D_2(K)}$.

Let us see an example of calculations of our invariants. Let $K$ be the knot as the boundary of the Seifert surface $R$ illustrated in Figure 4. This is the knot 0057 in Figure 7. We can easily compute that $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and the genus of $R$ is 2. Hence $K$ is a homologically fibered knot and $R$ is of minimal genus. The graph $G$ in the right hand side of Figure 4 is obtained from $R$ by a deformation retract. Thus $\pi_1(M_R) \cong \pi_1(S^3 - \overset{\circ}{N}(G))$. Then $\pi_1(M_R)$ has a presentation:

$$\left\langle\, z_1, z_2, \ldots, z_{10} \;\middle|\; z_1z_5z_6^{-1},\ z_2z_3z_4z_1,\ z_3z_9^{-1}z_5^{-1},\ z_7z_4z_8^{-1},\ z_8z_{10}z_6,\ z_2z_5z_7^{-1}z_5^{-1},\ z_9z_4z_{10}^{-1}z_4^{-1} \,\right\rangle.$$

The first 5 relations come from the vertices of $G$ and the last 2 relations come from the crossings of $G$. We can drop the last relation $z_9z_4z_{10}^{-1}z_4^{-1}$ because it is derived from the others.

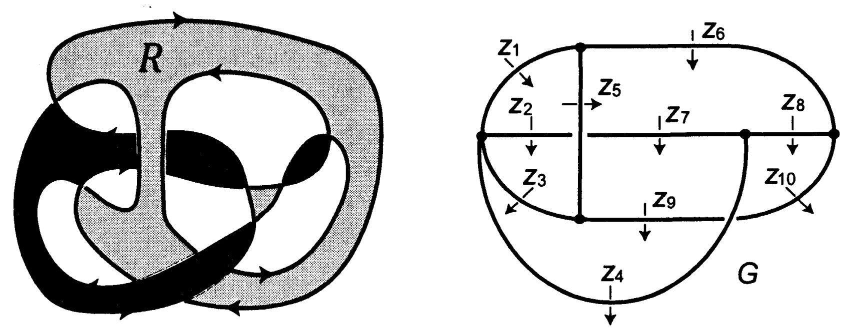


FIGURE 4

We take a spine of $R$ as in Figure 5, by which we can fix an identification of $\Sigma_{g,1}$ and $R$. A direct computation shows that

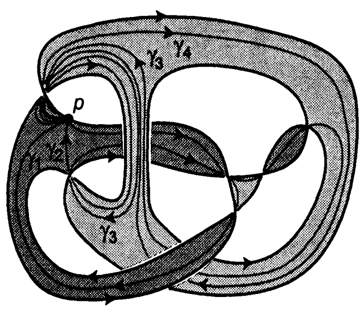


FIGURE 5

$$\begin{array}{llll} i_-(\gamma_1) = z_5z_1 & i_-(\gamma_2) = z_2^{-1} & i_-(\gamma_3) = z_5z_7^{-1}z_8^{-1}z_4^{-1} & i_-(\gamma_4) = z_4^{-1} \\ i_+(\gamma_1) = z_5 & i_+(\gamma_2) = z_6z_9 & i_+(\gamma_3) = z_6z_5^{-1}z_3z_5z_7^{-1}z_4^{-1}z_6^{-1} & i_+(\gamma_4) = z_6z_7z_6^{-1}. \end{array}$$

Here the darker color in $R$ is the +-side. Then, we obtain an admissible presentation of $\pi_1(M_R)$:

| | |
|---|---|
| Generators | $i_-(\gamma_1),\ldots,i_-(\gamma_4),\ z_1,\ldots,z_{10},\ i_+(\gamma_1),\ldots,i_+(\gamma_4)$ |
| Relations | $z_1z_5z_6^{-1},\ z_2z_3z_4z_1,\ z_3z_9^{-1}z_5^{-1},\ z_7z_4z_8^{-1},\ z_8z_{10}z_6,\ z_2z_5z_7^{-1}z_5^{-1},$ $i_-(\gamma_1)z_1^{-1}z_5^{-1},\ i_-(\gamma_2)z_2,\ i_-(\gamma_3)z_4z_8z_7z_5^{-1},\ i_-(\gamma_4)z_4,$ $i_+(\gamma_1)z_5^{-1},\ i_+(\gamma_2)z_9^{-1}z_6^{-1},\ i_+(\gamma_3)z_6z_4z_7z_5^{-1}z_3^{-1}z_5z_6^{-1},\ i_+(\gamma_4)z_6z_7^{-1}z_6^{-1}$ |

If we have an admissible presentation, we can use the program shown in Section 5. However, we here demonstrate a calculation by hand.

By sliding the edges $v_1$ and $v_2$ of $G$ as in Figure 6, we obtain a graph whose complement is clearly a genus 4 handlebody. This means that the complement of $G$ (and hence $M_R$) is homeomorphic to a genus 4 handlebody. Let $D_1,\ldots,D_4$ be the meridian disks of the handlebody as illustrated in the figure.

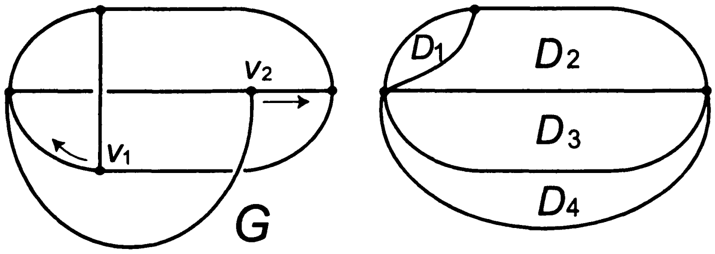


FIGURE 6

Then, $H_1(M_R)$ is the free abelian group generated by $t_i\,(i=1,\ldots,4)$ where $t_i$ corresponding to an oriented loop which intersects $D_i$ transversely in one point from the above to the down side in Figure 6 and is disjoint from $D_j\,(i\neq j)$.

We have the natural homomorphism $\pi_1(M_R)\to H_1(M_R)$ which maps

$$
\begin{array}{lllll}
z_1\mapsto t_1^{-1} & z_2\mapsto t_2t_3^{-1} & z_3\mapsto t_1t_2^{-1}t_3t_4^{-1} & z_4\mapsto t_4 & z_5\mapsto t_1t_2^{-1}\\
z_6\mapsto t_2^{-1} & z_7\mapsto t_2t_3^{-1} & z_8\mapsto t_2t_3^{-1}t_4 & z_9\mapsto t_3t_4^{-1} & z_{10}\mapsto t_3t_4^{-1}
\end{array}
$$

$$
\begin{array}{llll}
i_-(\gamma_1)\mapsto t_2^{-1} & i_-(\gamma_2)\mapsto t_2^{-1}t_3 & i_-(\gamma_3)\mapsto t_1t_2^{-3}t_3^2t_4^{-2} & i_-(\gamma_4)\mapsto t_4^{-1}\\
i_+(\gamma_1)\mapsto t_1t_2^{-1} & i_+(\gamma_2)\mapsto t_2^{-1}t_3t_4^{-1} & i_+(\gamma_3)\mapsto t_1t_2^{-2}t_3^2t_4^{-2} & i_+(\gamma_4)\mapsto t_2t_3^{-1}
\end{array}
$$

Under the bases $\langle[\gamma_1],[\gamma_2],[\gamma_3],[\gamma_4]\rangle$ of $H_1(\Sigma_{2,1})$ and $\langle t_1,t_2,t_3,t_4\rangle$ of $H_1(M_R)$, the induced maps $i_-,i_+$ are represented by

$$
S_-=\begin{pmatrix} 0 & 0 & 1 & 0\\ -1 & -1 & -3 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & -2 & -1\end{pmatrix},\qquad S_+=\begin{pmatrix} 1 & 0 & 1 & 0\\ -1 & -1 & -2 & 1\\ 0 & 1 & 2 & -1\\ 0 & -1 & -2 & 0\end{pmatrix}
$$

respectively. Note that $\det(I-t\,(S_+^{-1}S_-))=1-2t+3t^2-2t^3+t^4$ is the Alexander polynomial of $K$.

Since $M_R$ is homeomorphic to a handlebody, we have the following admissible presentation of $\pi_1(M_R)$ by setting $x_1:=z_1^{-1}, x_2=z_6^{-1}, x_3:=(z_6z_7)^{-1}$ and $x_4:=z_4$, which are mapped to $t_1$, $t_2$, $t_3$ and $t_4$ by the homomorphism $\pi_1(M_R)\to H_1(M_R)$.

| | |
|---|---|
| Generators | $i_-(\gamma_1),\ldots,i_-(\gamma_4),\ x_1,x_2,x_3,x_4,\ i_+(\gamma_1),\ldots,i_+(\gamma_4)$ |
| Relations | $i_-(\gamma_1)x_1x_2x_1^{-1},\ i_-(\gamma_2)x_1x_3^{-1}x_2x_1^{-1},\ i_-(\gamma_3)x_4x_2x_3^{-1}x_4x_2x_3^{-1}x_2x_1^{-1},\ i_-(\gamma_4)x_4,$ $i_+(\gamma_1)x_2x_1^{-1},\ i_+(\gamma_2)x_4x_3^{-1}x_2,\ i_+(\gamma_3)x_2^{-1}x_4x_2x_3^{-1}x_2x_1^{-1}x_4x_3^{-1}x_2,\ i_+(\gamma_4)x_2^{-1}x_3$ |

We write $r_1, \dots, r_8$ for these relations in order. Note that $\mathcal{K}_{H_1(M_R)}$ is isomorphic to the field of rational functions with variables $x_1, \dots, x_4$. Then we have:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{array}{c} \\ i_-(\gamma_1) \\ i_-(\gamma_2) \\ i_-(\gamma_3) \\ i_-(\gamma_4) \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ i_+(\gamma_1) \\ i_+(\gamma_2) \\ i_+(\gamma_3) \\ i_+(\gamma_4) \end{array} \begin{array}{c} \begin{array}{cccccccc} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

where $g_{ij} = \overline{\dfrac{\partial r_j}{\partial x_i}}$. Thus $\tau^+_{\rho_2}(M_R) = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix}$. As a torsion, it is equivalent to $\begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\ g_{25} & g_{26} & g_{27} & g_{28} \\ g_{35} & g_{36} & g_{37} & g_{38} \\ g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix}$, where

$$\begin{aligned} &g_{15} = -1, && g_{16} = 0, && g_{18} = 0, \\ &g_{25} = x_1^{-1}x_2, && g_{26} = x_2, && g_{28} = -x_3, \\ &g_{35} = 0, && g_{36} = -x_2, && g_{38} = x_3, \\ &g_{45} = 0, && g_{46} = x_2x_3^{-1}x_4, && g_{48} = 0, \end{aligned}$$

$$\begin{aligned} g_{17} &= -x_2x_3^{-1}x_4, \\ g_{27} &= x_2 + x_1^{-1}x_2^2x_3^{-1}x_4 + x_1^{-1}x_2^3x_3^{-2}x_4 - x_1^{-1}x_2^3x_3^{-2}x_4^2, \\ g_{37} &= -x_2 - x_1^{-1}x_2^2x_3^{-1}x_4, \\ g_{47} &= x_2x_3^{-1}x_4 + x_1^{-1}x_2^3x_3^{-2}x_4^2. \end{aligned}$$

Then we have:

$$\det(\tau^+_{\rho_2}(M_R)) = \det \begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\ g_{25} & g_{26} & g_{27} & g_{28} \\ g_{35} & g_{36} & g_{37} & g_{38} \\ g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix} = -\frac{x_2^3x_4^2}{x_1x_3^2}(x_2 - x_3 - x_2x_4).$$

The Magnus matrix $r_{\rho_2}(M_R)$ can be computed by the formula in Proposition 3.5 (2). However we omit here.

**Remark 4.1.** If we change bases of $H_1(\Sigma_{2,1}) \cong H_1(M_R)$ by

$$x_1 = \gamma_2^{-2}\gamma_3, \quad x_2 = \gamma_1^{-1}\gamma_2^{-2}\gamma_3, \quad x_3 = \gamma_1^{-1}\gamma_2^{-2}\gamma_3\gamma_4^{-1}, \quad x_4 = \gamma_2^{-1}\gamma_4^{-1},$$

where $\gamma_j$ denotes $i_+(\gamma_j)$, we have $\det(\tau^+_{\rho_2}(M_R)) = \dfrac{\gamma_3}{\gamma_1^2\gamma_2^5\gamma_4}(1+\gamma_2-\gamma_2\gamma_4)$. This expression is used in the program in Section 5.

## 5. MATHEMATICA PROGRAM

The following is a MATHEMATICA program which calculates the invariants discussed in the previous section.

```
h1Class = {};
h1Monodromy = {};
torsionMatrix = {};
magnusMatrix = {};

invariants[g_, z_, RELATIONS_] :=
  Module[{reindexedRel, h1Matrix, i, alex},
   GENUS = g;
   Ztotal = z;

   reindexedRel = Map[reindexing, RELATIONS, {2}];

   h1Matrix = -Map[Take[#, -2 GENUS] &, homologyComputation[reindexedRel]];
   h1Class =
    Join[Map[monomialExpression, h1Matrix],
     Table[ToExpression[ToString[SequenceForm["\[Gamma]", i]]], {i, 2 GENUS}]];
   Print["Homology classes of generators = ", h1Class // DisplayForm];

   h1Monodromy = Transpose[Take[h1Matrix, 2 GENUS]];
   Print["Homological monodromy = ", h1Monodromy // MatrixForm];

   alex = Transpose[makeAlexanderMatrix[reindexedRel]];
   torsionMatrix = Take[alex, 2 GENUS + Ztotal];
   Print["torsion matrix = ", torsionMatrix // MatrixForm];
   Print["det(torsion) = ", Expand[Det[torsionMatrix]]];

   magnusMatrix = Simplify[Transpose[
     Take[Transpose[-Drop[alex, 2 GENUS + Ztotal].Inverse[
           torsionMatrix]], 2 GENUS]]];
   Print["Magnus matrix = ", magnusMatrix // MatrixForm]
   ];

reindexing[num_] :=
  Module[{numString, sg},
    If[NumberQ[num], num + 2 GENUS*Sign[num],
     numString = ToString[num];
     sg = If[StringTake[numString, 1] == "-", 1, 0];
     If[StringTake[numString, {1 + sg}] == "m",
      ((-1)^sg)*ToExpression[StringDrop[numString, 1 + sg]],
      ((-1)^sg)*(ToExpression[StringDrop[numString, 1 + sg]] + 2 GENUS + Ztotal)]]
```

```
];

homologyComputation[rel_] :=
  Module[{i, j},
   RowReduce[Table[Count[rel[[i]], j] - Count[rel[[i]], -j],
   {i, 1, 2 GENUS + Ztotal}, {j, 1, 4 GENUS + Ztotal}]]];

monomialExpression[list_] :=
  Module[{i, prod = 1},
   For[i = 1, i <= 2 GENUS, i++,
    prod = prod*(ToExpression[ToString[SequenceForm["\[Gamma]", i]]]^list[[i]])];
   prod];

makeAlexanderMatrix[rel_] :=
  Module[{i, j},
   Table[foxDer[rel[[i]], j], {i, 1, Length[rel]}, {j, 1, 4 GENUS + Ztotal}]];

foxDer[word_, var_] :=
  Module[{entry = 0, i},
   For[i = 1, i <= Length[word], i++,
    Which[word[[i]] == var,
     entry = entry + (makeMonomial[Take[word, i - 1]]^(-1)),
     word[[i]] == -var,
     entry = entry - (makeMonomial[Take[word, i]]^(-1))]];
   entry];

makeMonomial[list_] :=
  Module[{prod = 1},
   For[i = 1, i <= Length[list], i++,
    prod = prod*(h1Class[[Abs[list[[i]]]]]^Sign[list[[i]]])];
   prod];
```

A computation by this program goes as follows. Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ with an admissible presentation

$$\langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \mid r_1, \ldots, r_{2g+l} \rangle$$

of $\pi_1(M)$. The main function in the program is `invariants` having three slots as the input. These slots correspond to the genus $g$, the number $l$ of $z$-generators and the list of relations. For each word in the relations, we make a list by replacing $i_-(\gamma_j)^{\pm 1}$, $z_j^{\pm 1}$ and $i_+(\gamma_j)^{\pm 1}$ by $\pm$`mj`, $\pm$`j` and $\pm$`pj`. By lining up them, we obtain the list of relations.

For example, the knot 0815 in Figure 7 has a minimal genus Seifert surface giving a sutured manifold whose fundamental group has the following admissible presentation:

| | |
|---|---|
| Generators | $i_-(\gamma_1), \ldots, i_-(\gamma_4),\ z_1, \ldots, z_{11},\ i_+(\gamma_1), \ldots, i_+(\gamma_4)$ |
| Relations | $z_1z_9z_6,\ z_1z_2^{-1}z_4^{-1},\ z_4z_{11}^{-1}z_5,\ z_{10}^{-1}z_5^{-1}z_6z_7z_8,\ z_8^{-1}z_6^{-1}z_9z_6,$ $z_7^{-1}z_6^{-1}z_3z_6,\ z_4z_3^{-1}z_4^{-1}z_{10},$ $i_-(\gamma_1)z_4z_3^{-1}z_4^{-1},\ i_-(\gamma_2)z_4z_{11},\ i_-(\gamma_3)z_9,\ i_-(\gamma_4)z_2^{-1}z_9^{-1},$ $i_+(\gamma_1)z_2^{-1}z_3^{-1}z_4^{-1},\ i_+(\gamma_2)z_{11}z_1,\ i_+(\gamma_3)z_9z_3^{-1}z_1,\ i_+(\gamma_4)z_9z_2^{-1}z_9^{-1}$ |

Then, the input is:

```
invariants[2, 11, {{1, 9, 6}, {1, -2,-4}, {4,-11, 5},
  {-10, -5, 6, 7, 8}, {-8, -6, 9, 6}, {-7, -6, 3, 6},
```

```
{4, -3, -4, 10}, {m1, 4, -3, -4}, {m2, 4, 11},
{m3, 9}, {m4, -2, -9}, {p1, -2, -3, -4}, {p2, 11, 1},
{p3, 9, -3, 1}, {p4, 9, -2, -9}}]
```

Then the function returns homology classes of generators in terms of $\gamma$j $:= i_+(\gamma_j) \in H_1(M_R)$, the homological monodromy matrix $\sigma(M_R)$, the torsion matrix $\tau^+_{\rho_2}(M_R)$ and the Magnus matrix $r_{\rho_2}(M_R)$. These data can be referred as the variables **h1Class**, **h1Monodromy**, **torsionMatrix** and **magnusMatrix**.

Using this program, we can easily check the calculations presented in [13] for 13 non-fibered homologically fibered knots with 12-crossings (Figure 7).

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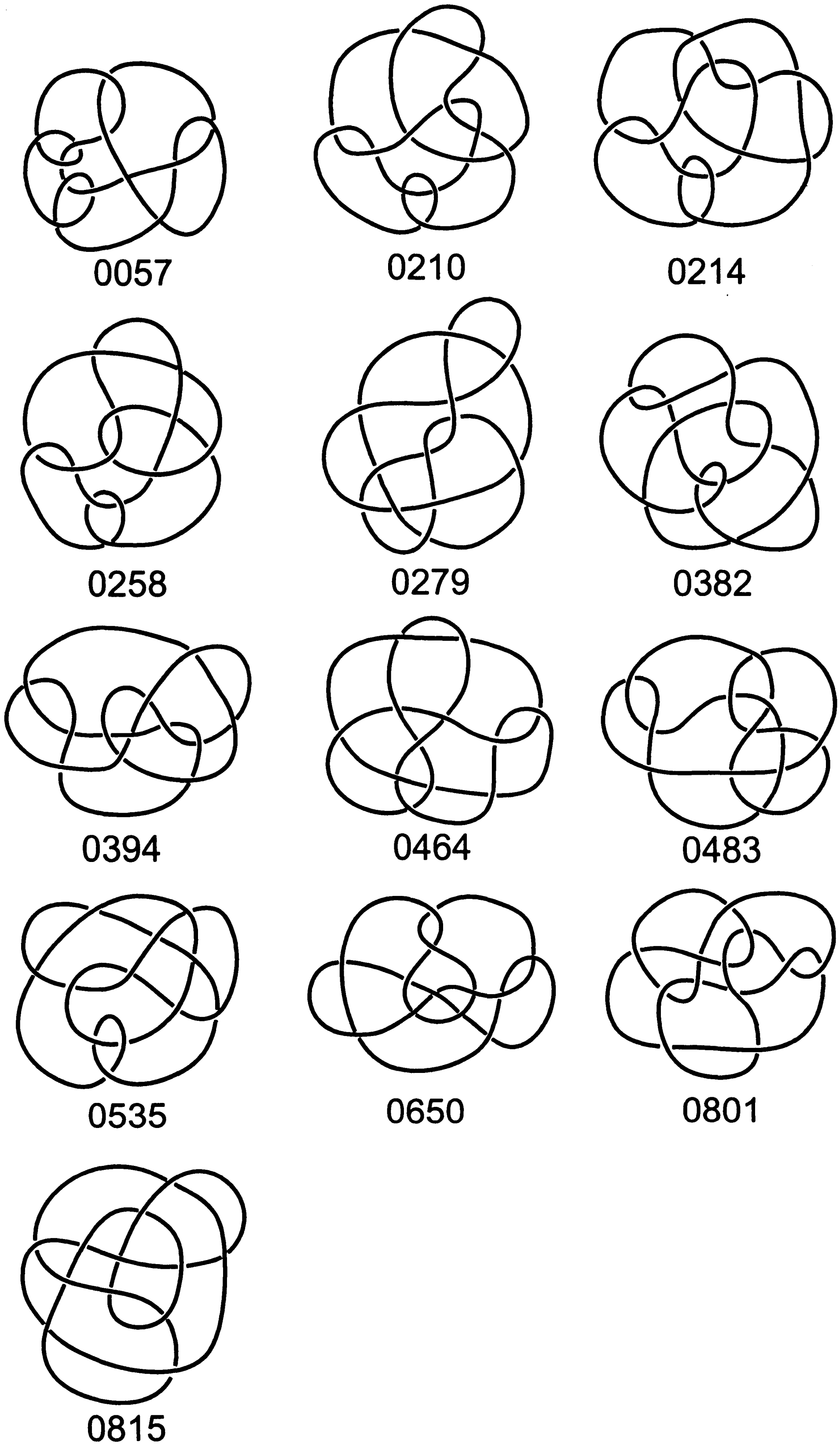


FIGURE 7. Non-fibered homologically fibered knots with 12-crossings